Schwarzian derivative and Numata Finsler structures

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Abstract

The flag curvature of the Numata Finsler structures is shown to admit a non-trivial prolongation to the one-dimensional case, revealing an unexpected link with the Schwarzian derivative of the diffeomorphisms associated with these Finsler structures.

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1 Finsler structures in a nutshell

1.1 Finsler metrics

A Finsler structure is a pair (M, F) where M is a smooth, n-dimensional, manifold and $F : TM \to \mathbb{R}^+$ a given function whose restriction to the slit tangent bundle $TM \backslash M = \{(x, y) \in TM \,|\, y \in T_xM \backslash \{0\}\}$ is strictly positive, smooth, and positively homogeneous of degree one, i.e., $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$; one furthermore demands that the $n \times n$ vertical Hessian matrix with entries $\mathrm{g}_{ij}(x, y) = \left(\frac{1}{2}F^2\right)_{y^iy^j}$ be positive definite, $(\mathrm{g}_{ij}) > 0$. See [1]. These quantities are (positively) homogeneous of degree zero, and the fundamental tensor

$$\mathrm{g} = \mathrm{g}_{ij}(x, y)dx^i \otimes dx^j \tag{1.1}$$

defines a *sphere's worth of Riemannian metrics* on each T_xM parametrized by the direction of y. See [2].

The distinguished "vector field"

$$\ell = \ell^i \frac{\partial}{\partial x^i}, \qquad \text{where} \qquad \ell^i(x, y) = \frac{y^i}{F(x, y)}, \tag{1.2}$$

actually a section of $\pi^*(TM)$ where $\pi : TM \backslash M \to M$ is the natural projection, is such that $\mathrm{g}(\ell, \ell) = 1$.

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There is a wealth of Finsler structures, apart from the special case of Riemannian structures (M, g) for which $F(x,y) = \sqrt{\mathrm{g}_{ij}(x)y^iy^j}$. For instance, the so-called Randers metrics

$$F(x,y) = \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i \tag{1.3}$$

satisfy all previous requirements if $a = a_{ij}(x)dx^i \otimes dx^j$ is a Riemann metric and if the 1-form $b = b_i(x)dx^i$ is such that $a^{ij}(x)b_i(x)b_j(x) < 1$ for all $x \in M$.

1.2 Flag curvature

Unlike the Riemannian case, there is no canonical linear Finsler connection on $\pi^*(TM)$. An example, though, is provided by the Chern connection $\omega^i_j = \Gamma^i_{jk}(x,y)dx^k$ which is uniquely defined by the following requirements [1]: (i) it is symmetric, $\Gamma^i_{jk} = \Gamma^i_{kj}$, and (ii) it *almost* transports the metric tensor, i.e., $dg_{ij} - \omega^k_i g_{jk} - \omega^k_j g_{ik} = 2C_{ijk}\delta y^k$, with $\delta y^i = dy^i + N^i_j dx^j$, where the $N^i_j(x,y) = \Gamma^i_{jk}y^k$ are the components of the non linear connection associated with the Chern connection, and the $C_{ijk}(x,y) = \frac{1}{2}\left(\mathrm{g}_{ij}\right)_{y^k}$ those of the Cartan tensor, specific to Finsler geometry.

Using the "horizontal covariant derivatives" $\delta/\delta x^i = \partial/\partial x^i - N^j_i \partial/\partial y^j$, one expresses the (horizontal-horizontal part of the) Chern curvature by

$$R_j{}^i{}_{kl} = \frac{\delta}{\delta x^k}\Gamma^i_{jl} + \Gamma^i_{mk}\Gamma^m_{jl} - (k \leftrightarrow l), \tag{1.4}$$

and the *flag curvature* (associated with the flag $\ell \wedge v$ defined by $v \in T_xM$) by

$$K(x,y,v) = \frac{R_{ik}v^iv^k}{\mathrm{g}(v,v) - \mathrm{g}(\ell,v)^2}, \qquad \text{where} \qquad R_{ik} = \ell^j R_{jik\ell}\,\ell^\ell. \tag{1.5}$$

One says that a Finsler structure is *of scalar curvature* if $K(x,y,v)$ does not depend on the vector v, i.e., if

$$R_{ik} = K(x,y)h_{ik}, \tag{1.6}$$

with $h_{ik} = \mathrm{g}_{ik} - \ell_i\ell_k$ the components of the "angular metric", where $\ell_i = \mathrm{g}_{ij}\ell^j (= F_{y^i})$. See [1, 2].

2 Numata Finsler structures

2.1 The Numata metric

Numata [4] has proved that metrics of the form $F(x,y) = \sqrt{q_{ij}(y)y^iy^j} + b_i(x)y^i$, on TM where $M \subset \mathbb{R}^n$, with $(q_{ij}) > 0$ and $db = 0$ are, indeed, of scalar curvature. See [2].

Of some interest is the special case $q_{ij} = \delta_{ij}$ and $b = df$ with $f \in C^\infty(M)$, viz.,

$$F(x,y) = \sqrt{\delta_{ij}y^iy^j} + f_{x^i}y^i, \tag{2.1}$$

where

$$M = \Big\{ x \in \mathbb{R}^n \;\Big|\; \sum_{i=1}^n f^2_{x^i} < 1 \Big\}. \tag{2.2}$$

The computation of the flag curvature of this particular Randers metric (1.3) can be found in [1] and yields

$$K(x,y) = \frac{3}{4}\frac{1}{F^4}\left(f_{x^i x^j} y^i y^j\right)^2 - \frac{1}{2}\frac{1}{F^3}\, f_{x^i x^j x^k} y^i y^j y^k. \tag{2.3}$$

2.2 Flag curvature & Schwarzian derivative

The expression (2.3) of the flag curvature of the Numata metric (2.1) holds for $n \geq 2$.

If $n = 1$, the left-hand side of (1.6) vanishes along with the curvature (1.4), while its right-hand vanishes as well since the angular metric has rank zero. For this particular dimension, Equation (1.6) trivially holds true, but tells, however, nothing about the flag curvature $K(x, y)$.

At this stage, it is worth noting that (2.3) indeed admits a prolongation to the one-dimensional case; it is therefore tempting to specialize its expression for $n = 1$.

Suppose, thus, that $M \subset S^1$ is a nonempty open subset (2.2), so that we have $TM\backslash M = T_+M \bigsqcup T_-M$, where $T_\pm M = M \times \mathbb{R}^\pm_*$. The metric (2.1) then reads

$$F(x,y) = |y| + f'(x)y, \tag{2.4}$$

using an affine coordinate, x, on S^1, with $-1 < f'(x) < +1$ (see (2.2)); its restrictions to $T_\pm M$ are given by $F_\pm(x,y) = \varphi'_\pm(x)y > 0$, where

$$\varphi'_\pm(x) = f'(x) \pm 1, \tag{2.5}$$

implying $\varphi_\pm \in \mathrm{Diff}_\pm(S^1)$, with $|\varphi'_\pm(x)| < 2$ (all $x \in M$).

The Numata metric (2.4) on T_+M, say, is thus associated, via (2.5), to orientation-preserving diffeomorphisms φ of S^1 such that $0 < \varphi'(x) < 2$ (all $x \in M$). Given such a $\varphi \in \mathrm{Diff}_+(S^1)$, the fundamental tensor (1.1) retains the form $\mathrm{g} = \varphi'(x)^2 dx^2$ and is, naturally, a Riemannian metric on M.

Rewriting Equation (2.3) for T_+M, and bearing in mind that $y = F(x,y)/\varphi'(x)$, we readily find that $K(x, y)$ is actually independent of y, namely

$$K(x) = -\frac{1}{2}\frac{1}{\varphi'(x)^2}\, S(\varphi)(x), \tag{2.6}$$

where

$$S(\varphi)(x) = \frac{\varphi'''(x)}{\varphi'(x)} - \frac{3}{2}\left(\frac{\varphi''(x)}{\varphi'(x)}\right)^2 \tag{2.7}$$

denotes the *Schwarzian derivative* [5] of the diffeomorphism φ of S^1. The argument clearly still holds, *mutatis mutandis*, for orientation-reversing diffeomorphisms of S^1.

We have thus proved the

Theorem 2.1. *The Numata Finsler structure* (M, F)*, with metric* F *given by (2.4) where* $M \subset S^1$ *is defined by (2.2), induces a Riemannian metric,* $\mathsf{g}(\varphi) = \varphi^*(dx^2)$*, where* $\varphi \in \mathrm{Diff}(S^1)$ *is as in (2.5). The flag curvature (2.3) admits a prolongation to this one-dimensional case and retains the form*

$$K = -\frac{1}{2}\frac{\mathsf{S}(\varphi)}{\mathsf{g}(\varphi)}, \tag{2.8}$$

where $\mathsf{S}(\varphi) = S(\varphi)(x)dx^2$ *is the Schwarzian quadratic differential of* $\varphi \in \mathrm{Diff}(S^1)$*.*

As an illustration, the one-dimensional Numata Finsler structures of constant flag curvature are associated, through (2.5), to the solutions φ of (2.8) for $K \in \mathbb{R}$, viz., $\varphi_{\pm}(x) = K^{-\frac{1}{2}} \arctan(K^{\frac{1}{2}}(ax+b)/(cx+d))$ where $a, b, c, d \in \mathbb{R}$ with $ad - bc = \pm 1$.

Let us mention another instance where the Schwarzian derivative is associated with curvature, namely the geometry of curves in Lorentzian surfaces of constant curvature [3].

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